A New Interval Finite Element Method: Computational Issues

Alba Sofi[1)] and Eugenia Romeo[2)]
[1)]*Department of Architecture and Territory (dArTe) and Inter-University Centre of Theoretical and Experimental Dynamics, University "Mediterranea" of Reggio Calabria, 89124, Reggio Calabria Italy, alba.sofi@unirc.it*
[2)]*Department of Civil, Energy, Environmental and Materials Engineering, University "Mediterranea" of Reggio Calabria, 89124 Reggio Calabria, Italy, eugenia.romeo@unirc.it*

Abstract: This paper deals with finite element analysis of linear structures with uncertain parameters modeled as interval variables. Uncertainties are handled by means of the *improved interval analysis via extra unitary interval* which enables to keep track of the dependencies between interval variables and thus reduce overestimation affecting both the assembly and solution phases of finite element procedures. Approximate explicit expressions of the bounds of the interval displacements are derived by applying the so-called *Interval Rational Series Expansion*. The computational efficiency of the method is enhanced by performing a preliminary sensitivity analysis to identify the most influential parameters on the selected response quantity. Numerical results are presented to demonstrate the accuracy and efficiency of the proposed procedure.

Keywords: finite element method, interval uncertainties, improved interval analysis, explicit expressions, sensitivity analysis, lower bound and upper bound

1. Introduction

Propagation of uncertainties affecting the design parameters has attracted the attention of several researchers over the last decades (Ayyub and Klir, 2006). Indeed, it is widely recognized that small variations of the input parameters may seriously affect the performance of an engineering system. In this context, the key issue is the selection of an appropriate mathematical model of uncertainty based on available empirical information. As known, the most widely used representation of uncertainties arising in engineering problems is the probabilistic one which is based on the concepts of random variable and random field characterized by appropriate Probability Density Functions (PDFs). In recent years, alternative uncertainty models based on non-probabilistic concepts (Elishakoff and Ohsaki, 2010; Corotis, 2015) have gained increasing importance in engineering applications. Such models turn out to be effective tools to describe and process uncertainties described by incomplete of fragmentary data, as happens in early design stages. Among non-probabilistic approaches, the interval model, originally developed on the basis of the interval analysis (Moore, 1996; Moore et al., 2009), has attracted the attention of many researchers mainly because of its simplicity and the small amount of required information. This model represents uncertainties as interval variables with given lower bound (LB) and upper bound (UB). No information is provided on the frequency of occurrence of values between the LB and UB.

The interval model of uncertainty has been extensively used in the context of finite element structural analysis giving rise to the so-called Interval Finite Element Method (IFEM). For a general overview of the state-of-art and recent advances in interval finite element analysis, readers are referred to (Moens and Vandepitte, 2005; Moens and Hanss, 2011). The main challenge to be faced in the application of the IFEM to real engineering problems is the so-called *dependency phenomenon* (Moore et al., 2009) which introduces conservatism both in the solution and matrix assembly phases. This phenomenon is mainly due to the inability of the *classical interval analysis* (*CIA*) to keep track of the dependency between interval variables. To limit overestimation of the interval solution, several versions of the IFEM have been developed, such as the element-by-element technique (Muhanna and Mullen, 2001), the sensitivity analysis method (Pownuk, 2004; Kreinovich et al., 2007) or the improvement of interval finite element analysis proposed by Degrauwe et al. (2010) based on *affine arithmetic*. Recently, the so-called *improved interval analysis via extra unitary interval* (*IIA via EUI*) (Muscolino and Sofi, 2012) has been introduced to perform interval structural analysis by taking into account dependencies between interval variables modeling uncertain physical properties. This is achieved by associating to each interval variable a particular unitary interval, the so-called *EUI*, which does not follow the rules of the *CIA*.

In this paper, a novel IFEM for the static analysis of linear-elastic structures with uncertain parameters is presented. Without loss of generality, Young's moduli of the FEs are modeled as independent interval variables, while applied loads are assumed to be deterministic. The key idea of the method is to handle interval variables by applying the *IIA via EUI* (Muscolino and Sofi, 2012) in order to reduce overestimation of the interval solution. Accordingly, an *EUI* is associated to each uncertain parameter and, therefore, to each FE. This allows to keep track of the dependencies between interval variables both in the assembly and solution phases of the finite element procedure. Then, approximate explicit expressions of the bounds of the interval nodal displacements are derived by applying the so-called *Interval Rational Series Expansion* (*IRSE*) (Muscolino and Sofi, 2013), recently proposed to evaluate the explicit inverse of an interval matrix with modifications. Finally, attention is focused on the computational efficiency of the presented IFEM. It is shown that the computational burden associated to the *IRSE* can be drastically reduced by retaining just the contribution of the most influential uncertain parameters. Such parameters are efficiently identified by performing a preliminary sensitivity analysis.

For validation purpose, a 3D cantilever beam with uncertain Young's modulus is analyzed. Both the accuracy and computational efficiency of the presented IFEM are investigated.

2. Interval Finite Element Formulation

2.1. INTERVAL MODEL OF UNCERTAINTY

The interval model of uncertainty describes the generic uncertain parameter as an interval variable (Moore, 1966) with given lower bound (LB) and upper bound (UB). It turns out to be very useful when only range information on the uncertain parameters is available, as happens in early design stages.

Let $\alpha_i^I = [\underline{\alpha}_i, \overline{\alpha}_i] \in \mathbb{IR}$ be an interval variable where $\mathbb{IR}$ is the set of all real interval numbers; the symbols $\underline{\alpha}_i$ and $\overline{\alpha}_i$ denote the LB and UB of the interval, respectively, while the apex I characterizes the interval variables. The i-th real interval variable $\alpha_i^I = [\underline{\alpha}_i, \overline{\alpha}_i]$ is such that $\underline{\alpha}_i \le \alpha_i \le \overline{\alpha}_i$ and it is characterized by the midpoint value (or mean), $\alpha_{0,i}$, and the deviation amplitude (or radius), $\Delta\alpha_i$, given by:

$$\alpha_{0,i} = \mathrm{mid}\{\alpha_i^I\} = \frac{\underline{\alpha}_i + \overline{\alpha}_i}{2}; \qquad \Delta\alpha_i = \frac{\overline{\alpha}_i - \underline{\alpha}_i}{2} \tag{1a,b}$$

where $\mathrm{mid}\{\bullet\}$ is an operator yielding the midpoint of the interval quantity between curly brackets.

In the framework of interval symbolism, a generic interval-valued function f and a generic interval-valued matrix function $\mathbf{A}$ of the interval vector $\boldsymbol{\alpha}^I \in \mathbb{IR}^r$, collecting the variables α_i^I, $(i=1,2,\ldots r)$, will be denoted in equivalent form, respectively, as:

$$f^I \equiv f(\boldsymbol{\alpha}^I) \Leftrightarrow f(\boldsymbol{\alpha}), \quad \boldsymbol{\alpha} \in \boldsymbol{\alpha}^I = [\underline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\alpha}}];$$
$$\mathbf{A}^I \equiv \mathbf{A}(\boldsymbol{\alpha}^I) \Leftrightarrow \mathbf{A}(\boldsymbol{\alpha}), \quad \boldsymbol{\alpha} \in \boldsymbol{\alpha}^I = [\underline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\alpha}}]. \tag{2a,b}$$

2.2. INTERVAL GLOBAL EQUILIBRIUM EQUATIONS

Let us consider a continuous elastic body which occupies the volume V bounded by the surface S in its undeformed state. The body is subjected to volume forces $\mathbf{b}(\mathbf{x})$ in V and surface forces $\mathbf{t}(\mathbf{x})$ on the loaded (or free) portion S_t of the boundary surface S, with $\mathbf{x} = [x_1 \quad x_2 \quad x_3]^{\mathrm{T}}$ denoting the position vector of a generic point referred to a Cartesian coordinate system $O(x_1, x_2, x_3)$; the displacements $\tilde{\mathbf{u}}(\mathbf{x})$ are imposed on the constrained portion S_u of S. The constitutive behavior of the material is linear-elastic isotropic. All input parameters are assumed to be known deterministically, except Young's modulus of the material which is modeled as an uncertain parameter.

Let the volume V of the body be subdivided into N_e finite elements (FEs). Young's modulus of each FE is modeled as an interval variable, i.e.:

$$E^{(i)}(\alpha_i^I) = E_0^{(i)}\left(1+\alpha_i^I\right), \qquad (i=1,2,\ldots,N^{(e)}) \tag{3}$$

where $\alpha_i^I = [\underline{\alpha}, \overline{\alpha}] \in \mathbb{IR}$ is a symmetric interval variable, i.e. characterized by a zero midpoint value $\alpha_{0,i} = 0$, which represents the dimensionless fluctuation around the nominal value $E_0^{(i)}$. Following the *improved interval analysis via extra unitary interval* (*IIA via EUI*) (Muscolino and Sofi, 2012), such fluctuation is herein expressed as (see Appendix):

$$\alpha_i^I = \Delta\alpha_i \hat{e}_i^I \tag{4}$$

where $\hat{e}_i^I = [-1,+1]$ is the *EUI*. In order to guarantee always positive values of the uncertain Young's modulus, the deviation amplitude of α_i^I must satisfy the condition $\Delta\alpha_i < 1$. Notice that an *EUI* is associated to each uncertain Young's modulus and, therefore, to each FE. This allows to link physical properties to the FEs and limit the overestimation due to the *dependency phenomenon* which typically affects both the assembly and solution phases of IFEMs based on the *CIA* (Moore et al., 2009).

Taking into account Eqs. (3) and (4), the elastic matrix of the i-th FE can be expressed as:

$$\mathbf{E}^{(i)}(\alpha_i^I) = \left(1+\Delta\alpha_i \hat{e}_i^I\right)\mathbf{E}_0^{(i)} \tag{5}$$

where $\mathbf{E}_0^{(i)}$ is the elastic matrix of the FE with nominal Young's modulus $E_0^{(i)}$.

Following the standard displacement-based FE formulation, the interval displacement field and the associated strain field within the i-th FE can be approximated as follows:

$$\mathbf{u}^{(i)}(\mathbf{x};\boldsymbol{\alpha}^I) = \mathbf{N}^{(i)}(\mathbf{x})\mathbf{d}^{(i)}(\boldsymbol{\alpha}^I) \tag{6}$$

and

$$\boldsymbol{\varepsilon}^{(i)}(\mathbf{x};\boldsymbol{\alpha}^I) = \mathbf{B}^{(i)}(\mathbf{x})\mathbf{d}^{(i)}(\boldsymbol{\alpha}^I) \tag{7}$$

where $\boldsymbol{\alpha}^I = [\underline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\alpha}}] \in \mathbb{IR}^{N_e}$ is the interval vector collecting the dimensionless fluctuations α_i^I, $(i=1,2,\ldots,N^{(e)})$, of Young's moduli of the N_e FEs; $\mathbf{N}^{(i)}(\mathbf{x})$ denotes the shape-function matrix; $\mathbf{B}^{(i)}(\mathbf{x})$ is the strain-displacement matrix; $\mathbf{d}^{(i)}(\boldsymbol{\alpha}^I)$ is the nodal displacement vector of the i-th FE which depends on the interval variables α_i^I.

Taking into account Eqs. (5) and (7), the stress field can be expressed by means of the constitutive equations as follows:

$$\boldsymbol{\sigma}^{(i)}(\mathbf{x};\boldsymbol{\alpha}^I) = \mathbf{E}^{(i)}(\alpha_i^I)\boldsymbol{\varepsilon}^{(i)}(\mathbf{x};\boldsymbol{\alpha}^I) = \left(1+\Delta\alpha_i \hat{e}_i^I\right)\mathbf{E}_0^{(i)}\mathbf{B}^{(i)}(\mathbf{x})\mathbf{d}^{(i)}(\boldsymbol{\alpha}^I). \tag{8}$$

Due to Young's modulus uncertainty, the element stiffness matrix turns out to be an interval matrix, defined as:

$$\mathbf{k}^{(i)}(\alpha_i^I) = \int_{V^{(i)}} \mathbf{B}^{(i)\mathrm{T}}(\mathbf{x})\mathbf{E}^{(i)}(\alpha_i^I)\mathbf{B}^{(i)}(\mathbf{x})\mathrm{d}V^{(i)} = \left(1+\Delta\alpha_i \hat{e}_i^I\right)\mathbf{k}_0^{(i)} \tag{9}$$

where $\mathbf{k}_0^{(i)} = \mathbf{k}^{(i)}(\alpha_i)\big|_{\boldsymbol{\alpha}=\mathbf{0}}$ is the nominal stiffness matrix.

The element force vector is not affected by uncertainties, i.e.:

$$\mathbf{f}^{(i)} = \int_{V^{(i)}} \mathbf{N}^{(i)\mathrm{T}}(\mathbf{x})\mathbf{b}(\mathbf{x})\mathrm{d}V^{(i)} + \int_{S_t^{(i)}} \mathbf{N}^{(i)\mathrm{T}}(\mathbf{x})\mathbf{t}(\mathbf{x})\mathrm{d}S^{(i)}. \tag{10}$$

It is worth emphasizing that, by applying the *IIA via EUI*, an *EUI* is associated to the stiffness matrix of each FE (see Eq.(9)). This allows to keep track of the dependencies between interval Young's moduli of the various FEs and thus perform standard assembly. Specifically, the assembly procedure yields the following set of linear interval equations governing the equilibrium of the FE model:

$$\mathbf{K}(\boldsymbol{\alpha}^I)\mathbf{U}(\boldsymbol{\alpha}^I) = \mathbf{F} \tag{11}$$

where $\mathbf{U}(\boldsymbol{\alpha}^I)$ is the interval vector collecting the n unknown nodal displacements, while

$$\mathbf{K}(\boldsymbol{\alpha}^I) \equiv \mathbf{K}^I = \sum_{i=1}^{N_e} \mathbf{L}^{(i)\mathrm{T}}\mathbf{k}^{(i)}(\alpha_i^I)\mathbf{L}^{(i)} \tag{12}$$

and

$$\mathbf{F} = \sum_{i=1}^{N_e} \mathbf{L}^{(i)\mathrm{T}}\mathbf{f}^{(i)} \tag{13}$$

are the interval global stiffness matrix of order $(n \times n)$ and the nodal force vector, respectively. Finally, in the previous equations, $\mathbf{L}^{(i)}$ denotes the connectivity matrix.

Taking into account Eq.(9), the interval global stiffness matrix can be rewritten as sum of the nominal value plus an interval deviation, i.e.:

$$\mathbf{K}^I = \mathbf{K}_0 + \sum_{i=1}^{N_e} \mathbf{L}^{(i)\mathrm{T}}\mathbf{k}_0^{(i)}\mathbf{L}^{(i)}\Delta\alpha_i \hat{e}_i^I \tag{14}$$

where $\mathbf{K}_0 = \mathbf{K}(\boldsymbol{\alpha})\big|_{\boldsymbol{\alpha}=\mathbf{0}}$ is the global nominal stiffness matrix

3. Bounds of the Solution

Within the interval framework, the solution of the set of interval equilibrium equations (11) involves the evaluation of the LB and UB vectors $\underline{\mathbf{U}}(\boldsymbol{\alpha})$ and $\overline{\mathbf{U}}(\boldsymbol{\alpha})$. To this aim, the knowledge of the explicit inverse of the interval stiffness matrix is crucial. Recently, the so-called *Interval Rational Series Expansion* (*IRSE*) (Muscolino and Sofi, 2013) has been derived as a modified explicit form of the Neumann series for evaluating the approximate inverse of an interval matrix with modifications. In the sequel, first the *IRSE* is summarized, then it is applied to derive approximate explicit bounds of the interval displacements.

3.1. Approximate Explicit solution: interval rational series expansion

The first step to apply the *IRSE* is the decomposition of the interval stiffness matrix as sum of the nominal value plus an interval deviation given by a superposition of rank-one matrices, i.e.:

$$\mathbf{K}(\boldsymbol{\alpha}^I) = \mathbf{K}_0 + \sum_{i=1}^{N_e} \alpha_i^I \, \mathbf{K}_i = \mathbf{K}_0 + \sum_{i=1}^{N_e} \sum_{\ell=1}^{p_i} \mathbf{s}_i^{(\ell)} \mathbf{v}_i^{(\ell)\mathrm{T}} \Delta\alpha_i \, \hat{e}_i^I \tag{15}$$

where $\mathbf{s}_i^{(\ell)}$ and $\mathbf{v}_i^{(\ell)}$ are column vectors and p_i is an integer number. The definition of these quantities depends on the kind of decomposition adopted for the matrices $\mathbf{K}_i$ (see e.g., Impollonia, 2006; Muscolino and Sofi, 2013; Muscolino et al., 2014).

By applying the *IRSE* truncated to first-order terms, the inverse of the interval stiffness matrix can be expressed in approximate explicit form as follows:

$$\left(\mathbf{K}^I\right)^{-1} \approx \mathbf{K}_0^{-1} - \sum_{i=1}^{N_e} \sum_{\ell=1}^{p_i} \frac{\Delta\alpha_i \, \hat{e}_i^I}{1 + d_{i\ell} \, \Delta\alpha_i \, \hat{e}_i^I} \mathbf{D}_{i\ell} \tag{16}$$

where

$$d_{i\ell} = \mathbf{v}_i^{(\ell)\mathrm{T}} \mathbf{K}_0^{-1} \mathbf{s}_i^{(\ell)}; \quad \mathbf{D}_{i\ell} = \mathbf{K}_0^{-1} \mathbf{s}_i^{(\ell)} \mathbf{v}_i^{(\ell)\mathrm{T}} \mathbf{K}_0^{-1}. \tag{17a,b}$$

Equation (16) provides the following approximate closed-form expression of the interval displacement vector:

$$\mathbf{U}^I = \left(\mathbf{K}^I\right)^{-1} \mathbf{F} \approx \mathbf{U}_0 - \sum_{i=1}^{N_e} \sum_{\ell=1}^{p_i} \frac{\Delta\alpha_i \, \hat{e}_i^I}{1 + d_{i\ell} \, \Delta\alpha_i \, \hat{e}_i^I} \mathbf{D}_{i\ell} \mathbf{F} \tag{18}$$

where $\mathbf{U}_0 = \mathbf{K}_0^{-1}\mathbf{F}$ is the solution pertaining to the nominal system.

3.2. Bounds of the interval displacements

In order to evaluate the lower bound and upper bound of the interval displacement vector, Eq. (18) can be conveniently rewritten in the following *affine form*:

$$\mathbf{U}^I = \mathbf{U}_0 + \sum_{i=1}^{N_e} \sum_{\ell=1}^{p_i} \left(a_{0,i\ell} - \Delta a_{i\ell} \hat{e}_i^I \right) \mathbf{D}_{i\ell} \mathbf{F} \tag{19}$$

where $a_{0,i\ell}$ and $\Delta a_{i\ell}$ are the midpoint and deviation amplitude of the generic term of the double summation in Eq. (18), given by:

$$a_{0,i\ell} = \frac{\Delta\alpha_i^2 d_{i\ell}}{1 - \left(\Delta\alpha_i d_{i\ell}\right)^2}; \qquad \Delta a_{i\ell} = \frac{\Delta\alpha_i}{1 - \left(\Delta\alpha_i d_{i\ell}\right)^2}. \tag{20a,b}$$

The argument $\Delta\alpha_i$ of the functions $a_{0,i\ell}$ and $\Delta a_{i\ell}$ is omitted for conciseness.

Based on Eq. (19) and applying the *IIA via EUI*, the following approximate explicit expressions of the LB and UB of the interval displacement vector $\mathbf{U}^I$ are obtained:

$$\underline{\mathbf{U}}(\boldsymbol{\alpha}) = \mathrm{mid}\left\{\mathbf{U}^I\right\} - \Delta\mathbf{U}(\boldsymbol{\alpha}); \quad \overline{\mathbf{U}}(\boldsymbol{\alpha}) = \mathrm{mid}\left\{\mathbf{U}^I\right\} + \Delta\mathbf{U}(\boldsymbol{\alpha}) \tag{21a,b}$$

where

$$\Delta\mathbf{U}(\boldsymbol{\alpha}) = \sum_{i=1}^{N_e} \left|\mathbf{R}_i\right| = \sum_{i=1}^{N_e} \left| \sum_{\ell=1}^{p_i} \Delta a_{i\ell}\, \mathbf{D}_{i\ell}\mathbf{F} \right| \tag{22}$$

is the deviation amplitude of $\mathbf{U}^I$ and the symbol $|\bullet|$ denotes absolute value component wise.

4. Sensitivity Analysis

The *IRSE* allows the evaluation of the interval displacement vector and the associated bounds in approximate closed-form. As known, the knowledge of the explicit dependence of the structural response on the design parameters is very useful for several purposes, such as reanalysis, optimization, sensitivity analysis, reliability analysis, etc. In this section, attention is focused on the evaluation of the sensitivities of the response with respect to the uncertain parameters. To this aim, let us recast Eq. (18) in the following form:

$$\mathbf{U}(\boldsymbol{\alpha}) = \mathbf{U}_0 - \sum_{i=1}^{N_e} \sum_{\ell=1}^{p_i} \frac{\alpha_i}{1 + d_{i\ell}\, \alpha_i} \mathbf{D}_{i\ell}\mathbf{F} \tag{23}$$

where $\alpha_i \in \alpha_i^I = \Delta\alpha_i\, \hat{e}_i^I$. Direct differentiation of Eq.(23) yields the following approximate explicit expression of the vector collecting the sensitivities of the interval displacements with respect to the i-th uncertain parameter:

$$\mathbf{s}_{\mathbf{U},i} = \left.\frac{\partial \mathbf{U}(\boldsymbol{\alpha})}{\partial \alpha_i}\right|_{\boldsymbol{\alpha}=\mathbf{0}} = -\sum_{\ell=1}^{p_i} \mathbf{D}_{i\ell}\mathbf{F}, \quad (i = 1, 2, \dots, N_e). \tag{24}$$

Sensitivities enable to predict how structural response is affected by a small change of the uncertain parameters. In the context of the presented IFEM, the knowledge of explicit response sensitivities can be exploited to enhance the computational efficiency of the *IRSE*. Indeed, the main drawback of the *IRSE* is that it involves a double summation (see Eq. (23)) which may be time consuming for real-sized structures. However, it can be readily inferred that not all the terms appearing in the *IRSE* approximation of the response (23) are equally important since each uncertain parameter has a different effect on the various DOFs of a structure. Based on this observation, we may perform a sensitivity analysis in order to detect those parameters which actually have a negligible influence on the response of a given DOF and then omit the corresponding contribution in the *IRSE*. This approach enables a drastic reduction of the computational time required by the *IRSE* which therefore can be efficiently used to analyze the response of structures with a large number of uncertain parameters.

The most influential parameters for each DOF can be identified by evaluating the so-called *coefficient of sensitivity* which provides a percentage measure of the global variability of the response with respect to its nominal value due to the generic uncertain parameter. Specifically, the *coefficient of sensitivity* of the nodal displacement $U_j(\boldsymbol{\alpha})$ with respect to the i-th parameter α_i can be defined as follows (Muscolino et al., 2014):

$$\beta_{i,U_j}(\%) = \left| \frac{1}{U_{0,j}} \left. \left(\frac{\partial U_j(\boldsymbol{\alpha})}{\partial \alpha_i} \right) \right|_{\boldsymbol{\alpha}=\mathbf{0}} \right| \Delta\alpha_i \times 100 \tag{25}$$

where $\Delta\alpha_i$ denotes the deviation amplitude of the dimensionless interval parameter $\alpha_i \in [-\Delta\alpha_i, \Delta\alpha_i]$; $U_{0,j} = U_j(\boldsymbol{\alpha})\big|_{\boldsymbol{\alpha}=\mathbf{0}}$ is the nominal value of the j-th displacement component.

For each DOF, the most crucial uncertain parameters are those characterized by higher values of the *coefficient of sensitivity*. Retaining only the terms of the *IRSE* corresponding to these parameters, say $r < N_e$, allows one to obtain simpler and more efficient analytical approximations of the response.

5. Numerical Application

For validation purpose, the response of a 3D cantilever beam with uncertain Young's modulus of the material (Figure 1) is analyzed. The beam is subjected on the top edge to a deterministic transversally distributed load $p_z = 10\ \mathrm{kN/m^2}$. The following geometrical properties are assumed: length $L = 5\mathrm{m}$ and rectangular cross section with width $b = 0.25\ \mathrm{m}$ and thickness $h = 0.5\ \mathrm{m}$. The nominal Young's modulus and Poisson ratio of the material are selected as $E_0 = 230\ \mathrm{GPa}$ and $\nu = 0.3$, respectively. A uniform FE mesh consisting of $N_e = 320$ eight-node brick elements is adopted. Young's modulus of each FE is modelled as an interval variable $E^{(i)}(\alpha_i^I) = E_0(1 + \Delta\alpha_i \hat{e}_i^I)$ with $\Delta\alpha_i = \Delta\alpha = 0.1$, $(i = 1, 2, \ldots, N_e)$.

The spectral decomposition of the nominal element stiffness matrix, $\mathbf{k}_0^{(i)} = \mathbf{k}^{(i)}(\alpha_i)\big|_{\boldsymbol{\alpha}=\mathbf{0}}$, is adopted in order to decompose the global stiffness matrix according to Eq. (15) and then apply the *IRSE*. In this case, $p_i = 18$ represents the number of deformation modes of the eight-node brick FE. The proposed IFEM is applied to evaluate the bounds of the interval displacement components, U_{zj}^I, $(j = 1, 2, \ldots, 20)$, along the z-axis of twenty selected nodes shown in Figure 1.

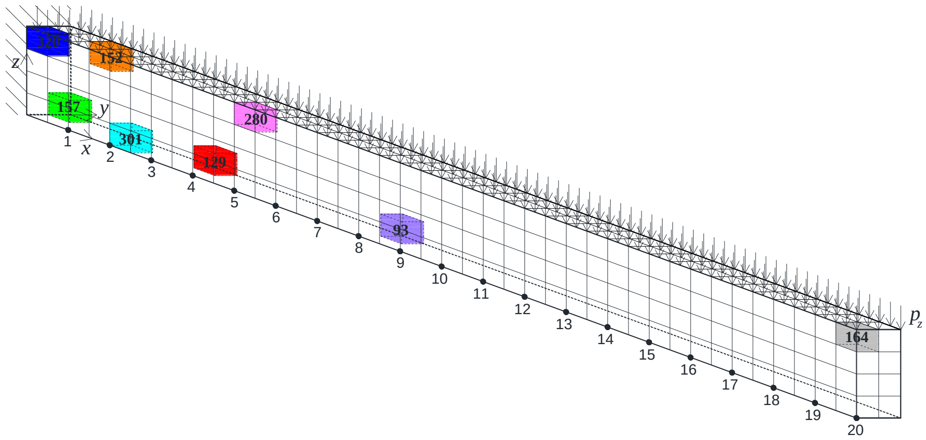

Figure 1. 3D cantilever beam with uncertain Young's modulus.

The accuracy of the estimates of displacement bounds could be assessed by comparisons with the exact bounds evaluated by applying a combinatorial procedure, known as *vertex method* (Dong and Shah, 1987). However, for the selected application, the *vertex method* is unfeasible since it requires 2^{320} deterministic analyses, as many as are the possible combinations of the endpoints of the uncertain parameters. For this reason, a comparison with the results obtained by applying a procedure based on sensitivity analysis (Pownuk, 2004; Kreinovich et al., 2007), herein referred to as *Sensitivity Method* (*SM*), is carried out.

First, the sensitivity of beam's response to the variability of Young's moduli of the $N_e = 320$ FEs is investigated by exploiting the closed-form expressions of displacement sensitivities derived from the *IRSE* (see Eqs. (24)). Figure 2 displays the sensitivities of the 20 nodal displacements $U_{zj}(\boldsymbol{\alpha})$, ($j = 1, 2, \ldots, 20$), of interest with respect to the fluctuations of the uncertain Young's moduli of eight selected FEs highlighted in Figure 1.

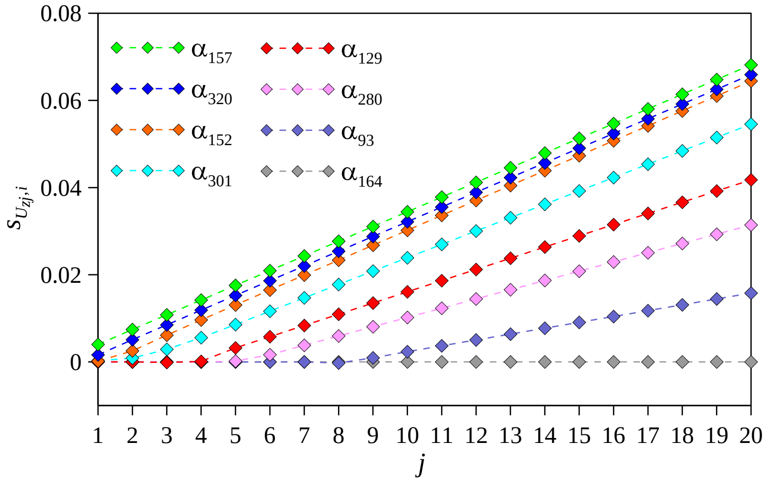

Figure 2. Sensitivities of the 20 selected nodal displacements of the cantilever beam with respect to the fluctuations of eight uncertain Young's moduli $E^{(i)}(\alpha_i^I) = E_0^{(i)}\left(1+\alpha_i^I\right)$, (see Figure 1).

In order to identify for each DOF the most crucial parameters, sensitivities can be ranked based on the corresponding *coefficients of sensitivity* reported in Figure 3. For instance, it can be observed that for all displacements the largest *coefficients of sensitivity* are those pertaining to the parameters α_{157} and α_{320} which, therefore, turn out to be the most influential among the eight selected parameters. Conversely, α_{93} and α_{164} are characterized by the smallest *coefficients of sensitivity* which means a less significant influence on the response. It is also observed that the displacement $U_{z1}(\boldsymbol{\alpha})$ is more sensitive to variations of the parameters α_{157} and α_{320} than the other displacements.

Based on the results of sensitivity analysis, a reduced form of the *IRSE* can be deduced by retaining for each DOF just the terms associated to the most influential uncertain parameters. Figure 4 shows a possible selection of the number r_j of significant terms of the *IRSE* for the 20 nodal displacements $U_{zj}(\boldsymbol{\alpha})$, $(j = 1, 2, \ldots, 20)$. Notice that the largest number of terms is needed to approximate $U_{z20}(\boldsymbol{\alpha})$ and it is still much smaller than the total number of uncertain parameters. This allows a substantial reduction of the computational effort.

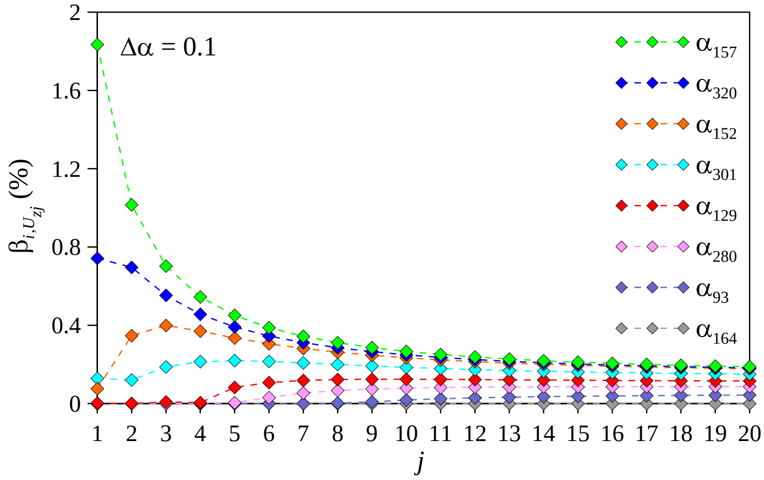

Figure 3. Coefficients of sensitivity of the 20 selected nodal displacements of the cantilever beam with respect to the fluctuations of eight uncertain Young's moduli $E^{(i)}(\alpha_i^I) = E_0^{(i)}\left(1+\alpha_i^I\right)$, $\alpha_i^I = \Delta\alpha\,\hat{e}_i^I$ with $\Delta\alpha = 0.1$ (see Figure 1).

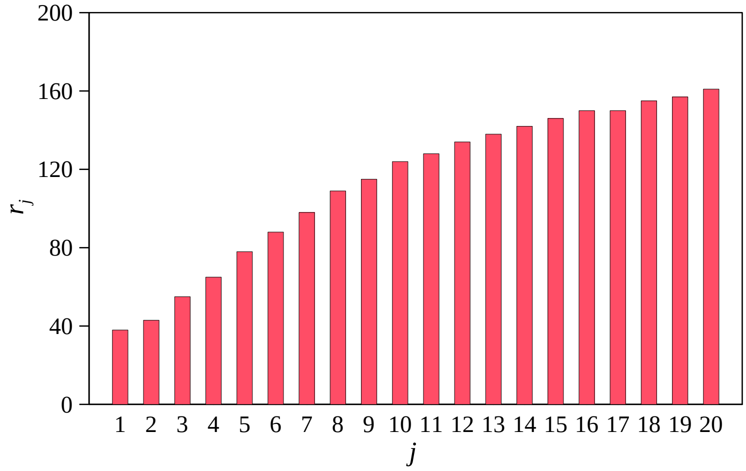

Figure 4. Number of terms r_j retained in the reduced *IRSE* for the 20 selected nodal displacements of the cantilever beam based on sensitivity analysis ($\Delta\alpha_i = 0.1$).

Figure 5 shows the comparison between the proposed bounds of the 20 selected nodal displacements obtained by applying the complete *IRSE* (with all terms retained) and a reduced *IRSE* involving just the number of terms reported in Figure 4. For validation purpose, the LB and UB of displacements provided by the *SM* are also reported. It can be seen that both the complete and reduced *IRSE*s yield approximate bounds in good agreement with those obtained by applying the *SM*. Furthermore, it is observed that the estimates pertaining to the reduced *IRSE* are very close to those provided by the complete *IRSE*. In particular the absolute percentage errors between the two approximations, reported in Figure 6, are always less than 0.5%. This demonstrates that terms omitted from the *IRSE* are actually negligible and the computational efficiency of the proposed IFEM can be greatly enhanced without affecting the accuracy of the results.

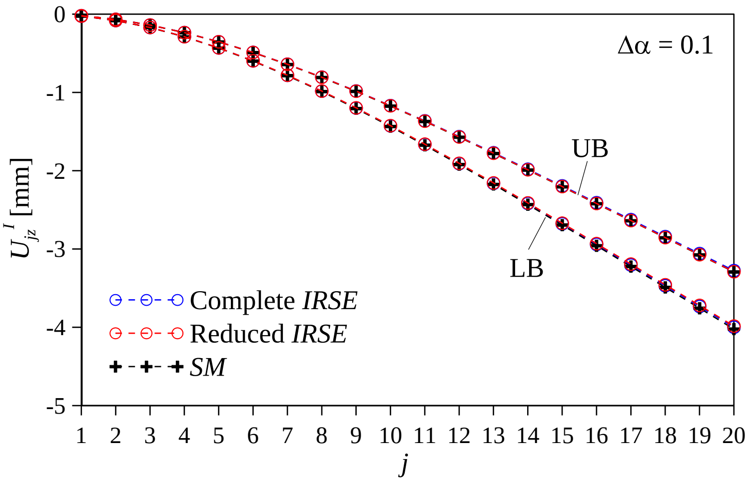

Figure 5. LB and UB of the 20 selected nodal displacements in the load direction of the cantilever beam with uncertain Young's moduli: comparison between the proposed bounds obtained by applying the reduced *IRSE* resulting from sensitivity analysis, the complete *IRSE* and the solution provided by the *SM* ($\Delta\alpha_i = 0.1$).

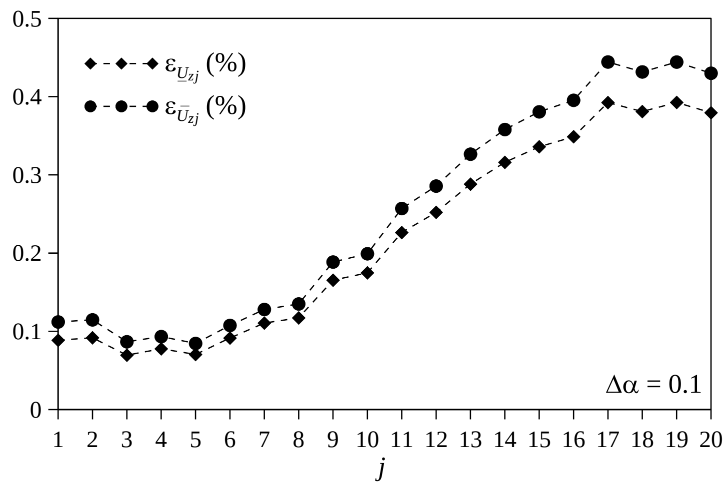

Figure 6. Absolute percentage errors affecting the LB and UB of the 20 selected nodal displacements of the cantilever beam provided by the reduced *IRSE* resulting from sensitivity analysis compared to the bounds yielded by the complete *IRSE* ($\Delta\alpha_i = 0.1$).

6. Conclusions

A novel Interval Finite Element Method (IFEM) for the static analysis of linear structures with uncertain parameters modeled as interval variables has been presented. The formulation relies on the use of the so-called *improved interval analysis*, recently introduced to limit the overestimation affecting the *classical interval analysis*. Accordingly, a particular unitary interval, called *extra unitary interval*, is associated to each uncertain parameter thus enabling to keep track of the dependencies between interval uncertainties in both the assembly and solution stages of the finite element procedure. The bounds of the interval displacements have been derived in approximate explicit form by applying the so-called *Interval Rational Series Expansion* (*IRSE*). Then, it has been shown that a preliminary sensitivity analysis of the response allows a drastic reduction of the computational effort required by the *IRSE* thus making the IFEM applicable to structures with a large number of uncertain parameters. To demonstrate the accuracy and efficiency of the proposed IFEM, a 3D cantilever beam with uncertain Young's modulus has been analyzed.

References

Ayyub, B.M. and G.J. Klir. Uncertainty Modeling and Analysis in Engineering and the Sciences. Chapman & Hall/CRC, Taylor & Francis Group, Boca Raton, 2006.

Corotis, R.B. An Overview of Uncertainty Concepts Related to Mechanical and Civil Engineering. *ASCE-ASME Journal of Risk and Uncertainty in Engineering Systems, Part B: Mechanical Engineering* 1(4) 040801, 2015.

Degrauwe, D., G. Lombaert and G. De Roeck. Improving Interval Analysis in Finite Element Calculations by Means of Affine Arithmetic. *Computers and Structures* 88: 247–254, 2010.

Dong, W. and H.C. Shah. Vertex Method for Computing Functions of Fuzzy Variables. *Fuzzy Sets and Systems* 24: 65 – 78, 1987.

Elishakoff, I. and M. Ohsaki. *Optimization and Anti-optimization of Structures under Uncertainty*. Imperial College Press, London, 2010.

Impollonia, N. A Method to Derive Approximate Explicit Solutions for Structural Mechanics Problems. *International Journal of Solids and Structures* 43: 7082–7098, 2006.

Kreinovich, V., J. Beck, C. Ferregut, A. Sanchez, G.R. Keller, M. Averill and S.A. Starks, Monte-Carlo-Type Techniques for Processing Interval Uncertainty, and their Potential Engineering Applications. *Reliable Computing* 13: 25–69, 2007.

Moens, D. and D. Vandepitte. A Survey of Non-Probabilistic Uncertainty Treatment in Finite Element Analysis. *Computer Methods in Applied Mechanics and Engineering* 194: 1527–1555, 2005.

Moens, D. and M. Hanss. Non-Probabilistic Finite Element Analysis for Parametric Uncertainty Treatment in Applied Mechanics: Recent Advances. *Finite Element and Analysis Design* 47: 4-16, 2011.

Moore, R. E. *Interval Analysis*. Prentice-Hall, Englewood Cliffs, 1966.

Moore, R. E., R. B. Kearfott and M. J. Cloud. *Introduction to Interval Analysis*. SIAM, Philadelphia, USA, 2009.

Muhanna, R. L. and R. L. Mullen. Uncertainty in Mechanics: Problems-Interval-Based Approach. *Journal Engineering Mechanics ASCE* 127: 557-566, 2001.

Muscolino, G. and A. Sofi. Stochastic Analysis of Structures with Uncertain-but-Bounded Parameters via Improved Interval Analysis. *Probabilistic Engineering Mechanics* 28: 152-163, 2012.

Muscolino, G., R. Santoro and A. Sofi. Explicit Sensitivities of the Response of Discretized Structures under Stationary Random Processes. *Probabilistic Engineering Mechanics* 35: 82-95, 2014.

Muscolino, G. and A. Sofi. Bounds for the Stationary Stochastic Response of Truss Structures with Uncertain-but-Bounded Parameters. *Mechanical Systems and Signal Processing* 37: 163-181, 2013.

Pownuk, A. Efficient Method of Solution of Large Scale Engineering Problems with Interval Parameters. In R.L. Muhanna and R.L. Mullen, editors, *Proceedings NSF Workshop on Reliable Engineering Computing*, 305–316, 2004.

Appendix-Improved Interval Analysis via Extra Unitary Interval

Interval computations based on the *classical interval analysis* (*CIA*) suffer from the overestimation due to the so-called *dependency phenomenon* which occurs when an expression contains multiple instances of one or more interval variables (Moore et al., 2009). In order to limit the conservatism due to this phenomenon and thus allow the applicability of the interval model of uncertainty to engineering problems, recently the *improved interval analysis via extra unitary interval* (*IIA via EUI*) has been proposed (Muscolino and Sofi, 2012). The key idea of the *IIA via EUI* is to express the i-th interval variable α_i^I in the following *affine form*:

$$\alpha_i^I = \alpha_{0,i} + \Delta\alpha_i \hat{e}_i^I \tag{A.1}$$

where $\hat{e}_i^I = [-1,+1]$ is the *EUI* which does not follow the rules of the *CIA*, i.e.:

$$\begin{aligned}
&\hat{e}_i^I - \hat{e}_i^I = 0; \qquad \hat{e}_i^I \times \hat{e}_i^I = \left(\hat{e}_i^I\right)^2 = [1,1];\\
&\hat{e}_i^I / \hat{e}_i^I = [1,1]; \quad \hat{e}_i^I \times \hat{e}_j^I = [-1,+1], \quad i \neq j;\\
&x_i \hat{e}_i^I \pm y_i \hat{e}_i^I = \left(x_i \pm y_i\right)\hat{e}_i^I;\\
&x_i \hat{e}_i^I \times y_i \hat{e}_i^I = x_i y_i \left(\hat{e}_i^I\right)^2 = x_i y_i [1,1].
\end{aligned} \tag{A.2a-f}$$

In these equations, $[1,1] = 1$ is the so-called unitary *thin interval* (Moore et al., 2009). It is worth emphasizing that the subscript i means that the *EUI*, $\hat{e}_i^I$, is associated to the i-th interval variable. By associating a different *EUI* to each interval variable, dependencies can be duly taken into account throughout calculations and the overestimation due to the *dependency phenomenon* can be drastically limited.